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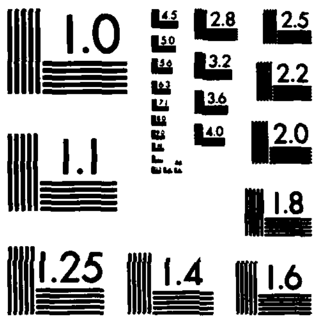

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TESTS FOR SPHERICITY UNDER CORRELATED
MULTIVARIATE REGRESSION EQUATIONS MODEL*

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Tests for Sphericity Under Correlated Multivariate Regression Equations Model

Shakuntala Sarkar and P. R. Krishnaiah

ABSTRACT

In this report, the authors considered some tests for sphericity of the error covariance matrix under a correlated multivariate regression equations ~~(CMRE)~~ model. Asymptotic distributions of the test statistics associated with the above procedures are also derived.

Keywords and Phrases: Sphericity, CMRE model, econometrics, multivariate regression.

1. INTRODUCTION

Extensive research has been done in the past on various problems connected with the classical multivariate regression model since this model plays a very important role in many problems like prediction. The multivariate regression model is nothing but a model with correlated univariate regression equations with a common design matrix. But, there are many situations when it is unrealistic to assume that the design matrices are the same. One such situation is when some of the observations on certain variables are missing. This situation has been dealt with in the statistical literature (e.g., see Srivastava (1966) and Trawinski (1961)) to a limited extent. Another situation is when the design matrices of different regression equations are not the same but none of the observations are missing. For example, the same independent variables may not be good to predict each and every dependent variable.

In the sequel, we will refer to the model based upon correlated univariate regression equations as the correlated regression equations (CRE) model. In econometric literature, the CRE model is known as seemingly unrelated regression equations model. Motivated by applications in economics, Revankar (1974, 1976), Srivastava (1970, 1973), Zellner (1962, 1963) and some other econometricians considered the problem of estimation of parameters under the CRE model when the underlying distribution is multivariate normal. Recently, Sarkar and Krishnaiah (1984) considered the problem of estimation of parameters under the CRE model when the underlying distribution is elliptically symmetric. Approximations to the distributions of the regression vector under the CRE model were discussed in Maekawa (1982) and Kariya and Maekawa (1982).

Kariya, Fujikoshi and Krishnaiah (1983) considered a model based upon two correlated multivariate regression equations and they refer to it as the correlated multivariate regression equations (CMRE) model. Under the above model, Kariya, Fujikoshi and Krishnaiah discussed various procedures for testing for the independence of the two sets of variables and also derived the asymptotic distributions of the statistics associated with the above test procedures. But, no work was done so far on tests for sphericity under the CMRE model.

In this paper, we discuss asymptotic distributions of various test statistics for sphericity under a CMRE model. The likelihood ratio test for sphericity was derived by Mauchly (1940) when the underlying distribution is a multivariate normal with unknown mean vector. Lee, Krishnaiah and Chang (1977) approximated certain powers of the likelihood ratio test statistic for sphericity with Pearson's type I distribution and the accuracy of this approximation is good for all practical purposes. If we know in advance about the structure of the covariance matrix, we can take advantage of this knowledge to propose more efficient estimates of the location parameters and better tests on these location parameters. So, it is quite important to investigate the structure of the covariance matrix of the underlying distribution and the independence of the two regression equations. The results derived in this paper are useful in studying the robustness of the LRT test for sphericity when the assumption of the same design matrix is violated under the usual multivariate regression model.

In Section 2, we give some preliminaries and state the problems that will be investigated in this paper. Throughout this paper, we use an estimate of the covariance matrix which is based upon the residuals connected with the regression equations. In Section 3, an asymptotic expression is obtained for the null distribution of the LRT-like test statistics for sphericity. When the design matrices of the regression equations are the same, the above test statistic reduces to the LRT test. For large samples, the asymptotic distribution of the LRT-like test is chi-square and it is the same as the asymptotic distribution of the LRT test statistic for sphericity when the design matrices of the regression equations are the same. But, if we take higher order terms, the expressions for the distributions will be different. In Section 4, we derive the asymptotic nonnull distribution of the LRT-like test for sphericity under the CMRE model under fixed alternatives. The expression obtained involves normal density and Hermite polynomials. The asymptotic distribution of the LRT-like test under local alternatives is given in Section 5. The expression derived in this section involves a linear combination of non-central chi-square variables. The results of Section 3-5 are derived under the assumption that the underlying distribution is multivariate normal. In Section 6, we have shown that the results of earlier sections remain true when the joint distribution of all the observations is elliptically contoured. Section 7 is devoted to a derivation of the moments of the estimate of the covariance matrix when the joint distribution of the observations on each variable is elliptically contoured but we do not assume that the joint distribution of all observations is elliptically contoured. In Section 8, it is shown that the asymptotic null distribution of the LRT-like test statistic is a linear combination of chi-square variables with one degree of freedom when the underlying distribution is as assumed in Section 7.

2. PRELIMINARIES

Consider two correlated regression equations

$$\begin{aligned} Y_1 &= X_1\theta_{11} + E_1 \\ Y_2 &= X_2\theta_{22} + E_2 \end{aligned} \tag{2.1}$$

where the design matrices X_1: $n \times r_1$, X_2: $n \times r_2$ are known and are assumed to be of full column rank. The matrices θ_{11}: $r_1 \times p_1$ and θ_{22}: $r_2 \times p_2$ of the parameters are unknown. We assume that the rows of $E = (E_1, E_2)$ are distributed independently as multivariate normal with mean vector $\underset{\sim}{0}$ and covariance matrix Σ, where

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \tag{2.2}$$

and Σ_{ij} is of order $p_i \times p_j$. An estimate of Σ is $\frac{1}{n}S$ where

$$S = \hat{E}'\hat{E} = \begin{pmatrix} Y_1'Q_1Y_1 & Y_1'Q_1Q_2Y_2 \\ Y_2'Q_2Q_1Y_1 & Y_2'Q_2Y_2 \end{pmatrix} \tag{2.3}$$

and

$$Q_i = I_n - X_i(X_i'X_i)^{-1}X_i' . \tag{2.4}$$

In this paper, we are interested in investigating the asymptotic null and nonnull distributions of various test statistics associated with testing the hypothesis H_0: $\Sigma = \sigma^2 I$ where σ^2 is an unknown constant.

We now discuss a representation of S which is used repeatedly in the sequel in deriving some of the distributions. This representation is due to Kariya, Fujikoshi and Krishnaiah (1983) and it is given in Lemma 2.1.

Consider the transformation

$$W_i = Z_i'Y_i = \begin{pmatrix} M_i \\ U_i \end{pmatrix} \tag{2.5}$$

where M_i is of order $(r_0-r_i) \times p_i$ and U_i is of order $(n-r_0) \times p_i$. Also Z_i: $(n-r_i) \times n$ satisfies $Z_i'Z_i = I_{n-r_i}, Z_iZ_i' = Q_i$ and is chosen in the following special way.

Let Z_0: $n \times (n-r_0)$ be a matrix satisfying

$$Q_0 = Z_0Z_0', \quad Z_0'Z_0 = I_{n_0}, \quad n_0 = n-r_0 \tag{2.6}$$

where $Q_0 = I-X(X'X)^+X'$, and $X = [X_1,X_2]$ where A^+ denotes the Penrose inverse of A. Further, let $\bar{Q}_j$ be the projection matrices onto $L(X) \cap L(Q_j)$, $(j = 1,2)$ where $L(A)$ denotes the column space of the matrix A. Also, let $\bar{Z}_j$ be a matrix satisfying

$$\bar{Q}_j = \bar{Z}_j\bar{Z}_j' \text{ and } \bar{Z}_j'\bar{Z}_j = I_{r_0-r_j} . \tag{2.7}$$

Then choose

$$Z_1 = (\bar{Z}_1, Z_0), \; Z_2 = (\bar{Z}_2, Z_0). \tag{2.8}$$

It is easy to verify that

$$Z_i'Z_i = I_{n_i} \text{ for } n_i = n-r_i \text{ and } Z_iZ_i' = Q_i,$$

where Q_i is given in (2.4). Note that under H_0, the rows of W_i are independently and identically distributed (i.i.d.) as multivariate normal with mean vector 0 and dispersion matrix $\sigma^2 I_{p_i}$. Also, W_1 and W_2 are distributed independent of each other.

From (2.3) and (2.4) we obtain

$$S = \begin{pmatrix} W_1'W_1 & W_1'Z_1'Z_2W_2 \\ W_2'Z_2'Z_1W_1 & W_2'W_2 \end{pmatrix} = G + B, \tag{2.9}$$

where

$$G = \begin{pmatrix} U_1' \\ U_2' \end{pmatrix} (U_1 \;\; U_2) \tag{2.10}$$

$$B = \begin{pmatrix} M_1'M_1 & M_1'KM_2 \\ M_2'K'M_1 & M_2'M_2 \end{pmatrix}, \; K = \bar{Z}_1'\bar{Z}_2. \tag{2.11}$$

These results can be summarized in the following lemma:

Lemma 2.1. The matrix S defined by (2.3) can be written as G+B, where G and B are defined by (2.10) and (2.11) respectively. Under H_0, G is distributed as $W_p(n_0, \sigma^2 I_p)$, where $p = p_1 + p_2$ and $n_0 = n - r_0$ and $\text{row}(M_i)$ is distributed as $N_{(r_0-r_i)p_i}[0, I_{r_0-r_i} \otimes \sigma^2 I_{p_i}]$, $i = 1,2,\ldots$ where M_i is defined by (2.5). Also, G and B are distributed independent of each other under H_0.

3. ASYMPTOTIC NULL DISTRIBUTION OF THE LRT-LIKE TEST STATISTIC

The hypothesis H_0 can be tested by using the statistic

$$\Lambda = \frac{|S|^{n/2}}{(\text{tr}S/p)^{np/2}} . \tag{3.1}$$

When $X_1 = X_2$, the above statistic is the likelihood ratio test statistic for sphericity. We will derive the null distribution of $\tilde{T} = a.T$ where

$$\begin{aligned} T &= -2 \log \Lambda \\ &= n[p \log \text{tr}\, S - \log |S| - p \log p] \end{aligned} \tag{3.2}$$

and $a = (n_0/n) \leq 1$, under the assumption that $K = \bar{Z}_1'\bar{Z}_2 = O(1)$ as $n_0 \to \infty$. Here we note that Λ is not the LRT test statistic when $X_1 \neq X_2$.

Let

$$V = \sqrt{n_0}\,\left(\frac{G}{n_0} - \sigma^2 I_p\right) \tag{3.3}$$

so that

$$G = n_0\left(\sigma^2 I_p + \frac{V}{\sqrt{n_0}}\right) . \tag{3.4}$$

So

$$S = n_0\sigma^2 (I+A), \tag{3.5}$$

where

$$A = \left(\frac{V}{\sqrt{n_0}\,\sigma^2} + \frac{B}{n_0\sigma^2}\right).$$

Now

$$\log |S| = p \log (n_0\sigma^2) + \log |I+A| \tag{3.6}$$

$$\log \operatorname{tr} S = \log (n_0\sigma^2) + \log p + \log \left(1 + \frac{\operatorname{tr} A}{p}\right). \tag{3.7}$$

From (3.2), (3.6) and (3.7) we obtain

$$\tilde{T} = T_0 + \frac{1}{\sqrt{n_0}}\,(T_1 + T_2), \tag{3.8}$$

where

$$T_0 = \frac{1}{2\sigma^4}\left[\operatorname{tr} V^2 - \frac{(\operatorname{tr} V)^2}{p}\right] \tag{3.9}$$

$$T_1 = \frac{1}{3\sigma^6}\left[\frac{(\operatorname{tr} V)^3}{p^2} - \operatorname{tr} V^3\right] \tag{3.10}$$

$$T_2 = \frac{1}{\sigma^4}\left[\operatorname{tr}(VB) - \frac{\operatorname{tr} V \operatorname{tr} B}{p}\right]. \tag{3.11}$$

The characteristic function of $\tilde{T}$ is

$$\phi(t) = E[e^{it\tilde{T}}]$$
$$= E[e^{itT_0}\{1+\frac{it}{\sqrt{n_0}}T_1\}] + E[e^{itT_0}\frac{it}{\sqrt{n_0}}T_2] + 0(n_0^{-1})$$
$$= \phi_1(t) + \phi_2(t) + 0(n_0^{-1}) \tag{3.12}$$

where

$$\phi_1(t) = E[e^{itT_0}\{1+\frac{it}{\sqrt{n_0}}T_1\}] \tag{3.13}$$

$$\phi_2(t) = E[e^{itT_0}\frac{it}{\sqrt{n_0}}T_2]\,. \tag{3.14}$$

Note that the characteristic function of $-\frac{2n_0}{n}\log\left[\frac{|G|}{(\frac{trG}{p})^p}\right]^{n/2}$ is $\phi_1(t)+0(n_0^{-1})$ where $G \sim W_p(n_0,\sigma^2 I_p)$. So we know that

$$\phi_1(t) = (1-2it)^{-f/2} + 0(n_0^{-1}) \tag{3.15}$$

where $f = (p(p+1)/2) - 1$.

Next we consider $\phi_2(t)$. Taking expectations with respect to M_i' s only yields

$$E[tr\,VB] = E[tr(V_{11}M_1'M_1 + V_{12}M_2'K'M_1 + V_{21}M_1'KM_2 + V_{22}M_2'M_2)]$$
$$= tr[V_{11}E(M_1'M_1) + V_{22}\,E(M_2'M_2)]$$
$$= \sigma^2(r_0-r_1)tr\,V_{11} + \sigma^2(r_0-r_2)\,trV_{22} \tag{3.16}$$

$$E[tr\,B] = E[tr\,M_1'M_1 + tr\,M_2'M_2]$$
$$= tr[\sigma^2(r_0-r_1)I_{p_1} + \sigma^2(r_0-r_2)I_{p_2}]$$
$$= [\sigma^2(r_0-r_1)p_1 + \sigma^2(r_0-r_2)p_2]. \tag{3.17}$$

From (3.14)

$$\phi_2(t) = E[e^{it T_0} \frac{it}{\sqrt{n_0}} \frac{(r_2 - r_1)}{p\sigma^2} \{p_2 \operatorname{tr} V_{11} - p_1 \operatorname{tr} V_{22}\}] \tag{3.18}$$

where

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}; \; V_{ij}: p_i \times p_j .$$

Now, note that the limiting distribution of $V = (v_{ij})$ in (3.3) is the distribution of $\bar{V} = (\bar{v}_{ij})$, where $\bar{v}_{ii} \sim N(0, 2\sigma^4)$, $\bar{v}_{ij} \sim N(0, \sigma^4)$ $i \neq j$ and $\bar{v}_{ij}$ $(i \leq j)$'s are all independent, and that the density f of V can be expressed as

$$f(V) = f_0(V) + \frac{1}{\sqrt{n_0}} f_1(V) + \frac{1}{n_0} f_2(V) + \ldots$$

where $f_0(V)$ is the p.d.f. of $\bar{V}$. Next, let

$$\begin{aligned} \underset{1\times p^2}{\underset{\sim}{v}^{*\prime}} &= (v_1^*, v_2^*, \ldots v_{p^2}^*), \\ &= (v_{11}, v_{22}, \ldots v_{pp}, v_{12}, \ldots v_{1p}, v_{21}, \ldots v_{2p}, \ldots v_{p1}, \ldots v_{pp-1}). \end{aligned} \tag{3.19}$$

Then the limiting distribution of $\underset{\sim}{v}^*$ is

$$N_{p^2}(0, D_\sigma) \tag{3.20}$$

where $D_\sigma = \mathrm{Diag}(2\sigma^4, \ldots, 2\sigma^4; \sigma^4, \ldots, \sigma^4)$.

Further note that

$$\begin{aligned} T_0 &= \frac{1}{2\sigma^4} [\sum_{i\neq j}\sum v_{ij}^2 + \frac{1}{p}\{(p-1)\sum v_{ii}^2 - \sum_{i\neq j} v_{ii} v_{jj}\}] \\ &= \frac{1}{2\sigma^4} \underset{\sim}{v}^{*\prime} A^* \underset{\sim}{v}^*, \quad \text{say} \end{aligned} \tag{3.21}$$

where A* depends on p.

So,

$$\phi_2(t) = C\frac{it}{\sqrt{n_0}}\int \exp\{-\frac{1}{2}v^{*\prime}(D_0^{-1} - \frac{it}{4}\Lambda)v^*\}[p_2\sum_{p_1+1}^{p_1} v_i^* - p_1\sum_{p_1+1}^{p} v_i^*]dv^* + O(n_0^{-1})$$

$$= 0 + O(n_0^{-1}) \tag{3.22}$$

Using (3.12), (3.15) and (3.22), we get

$$\phi(t) = (1-2it)^{-f/2} + O(n_0^{-1}) \tag{3.23}$$

where $f = (p(p+1)/2) - 1$. Now, inverting the right side of (3.23) yields the following expression for the asymptotic distribution of $\tilde{T}$:

$$\Pr(\tilde{T} \le x) = \Pr(\chi_f^2 \le x) + O(n_0^{-1}),$$

where $f = (p(p+1)/2) - 1$

4. ASYMPTOTIC NON-NULL DISTRIBUTION OF THE LRT-LIKE TEST STATISTIC UNDER FIXED ALTERNATIVE

Let us consider the alternative H_1: not H_0. Since the test statistic Λ is a function of the eigenvalues of S, we can assume, without loss of generality, that

$$\Sigma = D_\lambda = \mathrm{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_p). \tag{4.1}$$

Also, G, B, M_i's, n_0 are defined as in Section 2.2.

Let

$$V = \sqrt{n_0}\,(\frac{G}{n_0} - D_\lambda). \tag{4.2}$$

Under H_1, $G \sim W_p(n_0, D_\lambda)$. Now

$$S = n_0(D_\lambda + \frac{V}{\sqrt{n_0}} + \frac{B}{n_0})$$

$$= n_0 D_\lambda (I + A), \tag{4.3}$$

where

$$A = \left(\frac{1}{\sqrt{n_0}} D_\lambda^{-1} V + \frac{1}{n_0} D_\lambda^{-1} B\right)$$

$$\log |S| = p \log n_0 + \sum_{i=1}^{p} \log \lambda_i + \log |I + A|$$

$$\operatorname{tr} S = n_0 \left(\sum_{i=1}^{p} \lambda_i + \operatorname{tr} C \right), \quad C = \left(\frac{V}{\sqrt{n_0}} + \frac{B}{n_0}\right)$$

$$= n_0 p\bar{\lambda}\left(1 + \frac{\operatorname{tr} C}{p\bar{\lambda}}\right), \tag{4.4}$$

where $p\bar{\lambda} = \lambda_1 + \ldots + \lambda_p$. Also,

$$p \log \operatorname{tr} S = p \log n_0 + p \log p + p \log \bar{\lambda} + p \log \left(1 + \frac{\operatorname{tr} C}{p\bar{\lambda}}\right). \tag{4.5}$$

Hence

$$T = n\left[\log \frac{(\bar{\lambda})^p}{\prod_{i=1}^{p} \lambda_i} + p \log \left(1 + \frac{\operatorname{tr} C}{p\bar{\lambda}}\right) - \log |I+A|\right]. \tag{4.6}$$

Now, let

$$\tilde{\tilde{T}} = \sqrt{n_0}\left[\frac{1}{n} T - \log \frac{\bar{\lambda}^p}{\prod_{i=1}^{p} \lambda_i}\right]$$

$$= \sqrt{n_0}\left[p \log \left(1 + \frac{\operatorname{tr} C}{p\bar{\lambda}}\right) - \log |I+A|\right]$$

$$= T_0 + \frac{1}{\sqrt{n_0}} (T_1 + T_2) + O(n_0^{-1}), \tag{4.7}$$

where

$$T_0 = \sum_{i=1}^{p} \left(\frac{1}{\bar{\lambda}} - \frac{1}{\lambda_i}\right) v_{ii} \tag{4.8}$$

$$T_1 = -2 \sum_{i}^{p} \sum_{j}^{p} \frac{v_{ij}^2}{\lambda_i} - \frac{1}{2} \frac{\left(\sum^{p} v_{ii}\right)^2}{p\bar{\lambda}^2} \tag{4.9}$$

$$T_2 = \sum_{i=1}^{p} \left(\frac{1}{\lambda} - \frac{1}{\lambda_i}\right) b_{ii}. \tag{4.10}$$

The characteristic function of $\tilde{\tilde{T}}$ is

$$\psi(t) = \psi_1(t) + \psi_2(t) + 0(n_o^{-1}) \tag{4.11}$$

where

$$\psi_1(t) = E[e^{it\,T_0}(1 + \frac{it}{\sqrt{n_0}} T_1)] \tag{4.12}$$

$$\psi_2(t) = E[e^{it\,T_0} \frac{it}{\sqrt{n_0}} T_2]. \tag{4.13}$$

Defining $\underset{p^2\times 1}{\underset{\sim}{v}^*}$ as in (3.19), we see that the density of $\underset{\sim}{v}^*$ is

$N_{p^2}(\underset{\sim}{0},\Delta) + 0\ (n_0^{-1/2})$, where

$$\Delta = \mathrm{Diag}(2\lambda_1^2,\dots,2\lambda_p^2,\lambda_1\lambda_2,\dots\lambda_1\lambda_p,\lambda_2\lambda_1,\dots,\lambda_p\lambda_{p-1}). \tag{4.14}$$

Let

$$a_i = \left(\frac{1}{\lambda} - \frac{1}{\lambda_i}\right),\ i = 1(1)p \tag{4.15}$$

and

$$\underset{1\times p^2}{\underset{\sim}{a}'} = (a_1,a_2,\dots,a_p,0,0,\dots,0). \tag{4.16}$$

From (4.8), we know that $T_0 = \underset{\sim}{a}'\underset{\sim}{v}^*$. Also from (2.9), T_1 can be written as $\underset{\sim}{v}^{*\prime}\Omega\underset{\sim}{v}^*$, where elements of Ω: $p^2 \times p^2$ depend on $\lambda_1,\dots,\lambda_p$ and p. Then from (4.12),

$$\begin{aligned}\psi_1(t) &= E[e^{it\underset{\sim}{a}'\underset{\sim}{v}^*}(1 + \frac{it}{\sqrt{n_0}}\underset{\sim}{v}^{*\prime}\Omega\underset{\sim}{v}^*)] \\ &= e^{-\frac{t^2}{2}\underset{\sim}{a}'\Delta\underset{\sim}{a}}\left[1 + \frac{it}{\sqrt{n_0}}\,\mathrm{tr}\,(\Omega\Delta) + \frac{(it)^3}{\sqrt{n_0}}\underset{\sim}{a}'\Delta\Omega\Delta\underset{\sim}{a} + \frac{(it)^3}{\sqrt{n_0}}\frac{4\sum_j^p (a_j\lambda_j)^3}{3}\right]\end{aligned} \tag{4.17}$$

We now consider $\psi_2(t)$. Taking conditional expectation with respect to M_i's only yields

$$
\begin{aligned}
E_M(T_2) &= E[\sum_{i=1}^{p_1} a_i b_{ii} + \sum_{i=p_1+1}^{p} a_i b_{ii}] \\
&= \sum_{i=1}^{p_1} a_i (r_0 - r_1)\lambda_i + \sum_{i=p_1+1}^{p} a_i (r_0 - r_2)\lambda_i \\
&= (r_0 - r_1)\sum_{i=1}^{p_1} a_i\lambda_i + (r_0 - r_2)\sum_{i=p_1+1}^{p} a_i\lambda_i \\
&= K_1(r_0, r_1, r_2, a_i\text{'s}, \lambda_i\text{'s}, p_1, p_2) \\
&\equiv K_1, \quad \text{say}\ . \qquad (4.18)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\psi_2(t) &= \frac{it}{\sqrt{n_0}} K_1 \, E[e^{it\,\underset{\sim}{a}'\,\underset{\sim}{v}^*}] \\
&= \frac{it}{\sqrt{n_0}} K_1 \, e^{-\frac{t^2}{2}\underset{\sim}{a}'\Delta\underset{\sim}{a}}\ . \qquad (4.19)
\end{aligned}
$$

Finally, from (4.11), (4.17) and (4.19). we have

$$
\psi(t) = e^{-\frac{t^2\tau^2}{2}}[1 + \frac{it}{\sqrt{n_0}} g_1 + \frac{(it)^3}{\sqrt{n_0}} g_3] + 0(n_0^{-1}) \qquad (4.20)
$$

where

$$
\left.
\begin{aligned}
\tau^2 &= \underset{\sim}{a}'\Delta\underset{\sim}{a} = 2\sum_{i=1}^{p}(1 - \frac{\lambda_i}{\bar{\lambda}})^2 \\
g_1 &= K_1 + \mathrm{tr}(\Omega\Delta) \\
g_3 &= \underset{\sim}{a}'\Delta\Omega\Delta\underset{\sim}{a} + \frac{4}{3}\sum_{j=1}^{p} a_j^3\lambda_j^3\ .
\end{aligned}
\right\} \qquad (4.21)
$$

Note that under H_1, $\iota^2 \neq 0$, so that inverting the rightside of (4.20), we have the following theorem for the asymptotic distribution of $\tilde{\tilde{T}}$.

<u>Theorem 4.1</u>. The distribution function of $T^* = \tilde{\tilde{T}}/\tau = \sqrt{n_0}\,(\frac{1}{n}T - \log\frac{\bar{\lambda}^p}{\Pi\lambda_i})/\tau$ under H_1 can be expanded for large n as

$$\Pr[T^* < x] = \Phi(x) - \frac{1}{\sqrt{n_0}}\,[g_1\ \phi^{(1)}(x)/\tau + g_3\,\frac{\phi^{(3)}(x)}{\tau^3}] + O(n_0^{-1})$$

where $\phi^{(j)}(x)$ is the j^{th} derivative of the standard normal distribution function $\Phi(x)$; and g_1, g_2 and τ are given by (4.21).

5. ASYMPTOTIC NON-NULL DISTRIBUTION OF THE LRT-LIKE TEST UNDER LOCAL ALTERNATIVES

We assume the same structure of Σ as in (4.1), but we consider local alternatives

$$H_\theta:\ \lambda_i = \lambda + \frac{\theta_i}{\sqrt{n_0}},\quad i = 1(1)p \tag{5.1}$$

where θ_i's are not all equal. Thus, under H_θ,

$$D_\lambda = \lambda I_p + \frac{D_\theta}{\sqrt{n_0}}\ . \tag{5.2}$$

where $D_\theta = \mathrm{Diag}(\theta_1, \theta_2, \ldots, \theta_p)$. Under H_θ,

$$G \sim W_p[n_0, (\lambda I_p + \frac{D_\theta}{\sqrt{n_0}})]. \tag{5.3}$$

Define V as in (4.2), where D_λ is given by (5.2). We then have

$$S = n_0\lambda(I_p + A), \tag{5.4}$$

where $A = ((V+D_\theta)/\lambda\sqrt{n_0}) + (B/\lambda\, n_0)$. Expanding $\log|S|$ and $\log \mathrm{tr}\, S$ in the same way as before, we get

$$\tilde{T} = -2a \log \Lambda, \quad a = \frac{n_0}{n} \le 1$$

$$= T_0 + \frac{1}{\sqrt{n_0}}(T_1 + T_2), \tag{5.5}$$

where

$$T_0 = \frac{1}{2\lambda^2}\left[\mathrm{tr}(V+D_0)^2 - \frac{(\mathrm{tr}\,V + \sum_1^p \theta_i)^2}{p}\right] \tag{5.6}$$

$$T_1 = \frac{1}{3\lambda^3}\left[\frac{(\mathrm{tr}\,V + \sum_{i=1}^p \theta_i)^3}{p^2} - \mathrm{tr}(V+D_\theta)^3\right] \tag{5.7}$$

$$T_2 = \frac{1}{\lambda^2}\left[\mathrm{tr}\;B(V+D_\theta) - \frac{\mathrm{tr}\;B(\mathrm{tr}\,V + \sum_1^p \theta_i)}{p}\right]. \tag{5.8}$$

The characteristic function of $\tilde{T}$, as before, can be written as

$$\phi(t) = \phi_1(t) + \phi_2(t) + 0(n_0^{-1}), \tag{5.9}$$

where $\phi_1(t)$, $\phi_2(t)$ have the same expressions as in (3.13), (3.14) respectively.

Now, $\phi_1(t) + 0(n_0^{-1})$ is the characteristic function of $-2\, n_0/n \log\left[\frac{|G|}{(\frac{\mathrm{tr}\,G}{p})^p}\right]^{n/2}$, where under H_θ, $G \sim W_p[n_0, (\lambda I_p + \frac{D_\theta}{\sqrt{n_0}})]$, and hence $\phi_1(t)$ is known as (see Fujikoshi, 1981)

$$\phi_1(t) = \psi_f(t;\delta^2/\lambda^2)\left[1 + \frac{1}{\sqrt{n_0}}\sum_{j=0}^{2} \lambda^{-3} b_j (1-2it)^{-j}\right] + 0(n_0^{-1}) \tag{5.10}$$

where $\psi_f(t;\Delta)$ is the characteristic function of a noncentral χ^2 variable with f d.f. and noncentrality parameter δ^2/λ^2, and

$$f = \frac{(p+2)(p-1)}{2},$$

$$\delta^2 = \frac{1}{4}[\mathrm{tr}\;D_0^2 - p^{-1}(\mathrm{tr}\;D_0)^2]$$

$$b_0 = \frac{1}{6}[2\,\mathrm{tr}\,D_\theta^3 - 3p^{-1}(\mathrm{tr}\;D_\theta)\mathrm{tr}\;D_\theta^2 + p^{-2}(\mathrm{tr}\;D_\theta)^3]$$

$$b_1 = \frac{1}{2}\{-\operatorname{tr} D_\theta^3 + 2p^{-1}(\operatorname{tr} D_\theta)\operatorname{tr} D_\theta^2 - p^{-2}(\operatorname{tr} D_\theta)^3\}$$

$$b_2 = \frac{1}{6}\{\operatorname{tr} D_\theta^3 - 3p^{-1}(\operatorname{tr} D_\theta)\operatorname{tr} D_\theta^2 + 2p^{-2}(\operatorname{tr} D_\theta)^3\}.$$

Next consider $\phi_2(t)$.

Taking conditional expectations with respect to M_i's only we have

$$E_M(T_2) = \frac{(r_2-r_1)}{p\lambda}[p_2 \operatorname{tr}\tilde{V}_{11} - p_1 \operatorname{tr}\tilde{V}_{22}] + O(n_0^{-1/2}) \qquad (5.11)$$

where

$$\tilde{V} = (V+D_\theta) = \begin{pmatrix} \tilde{V}_{11} & \tilde{V}_{12} \\ \tilde{V}_{21} & \tilde{V}_{22} \end{pmatrix}$$

$\tilde{V}_{ij}$: $p_i \times p_j$. Writing $\tilde{V} = (\tilde{v}_{ij})$, see that

$$\tilde{v}_{ii} = v_{ii} + \theta_i, \quad i = 1,2,\ldots,p$$

$$\tilde{v}_{ij} = v_{ij}, \quad i \neq j.$$

As before, we write $T_0 = \frac{1}{2\lambda^2}\tilde{\underset{\sim}{v}}^{*\prime} A_0 \tilde{\underset{\sim}{v}}^{*}$, where

$$\underset{1\times p^2}{\tilde{\underset{\sim}{v}}^{*\prime}} = (\tilde{v}_1^*, \tilde{v}_2^*, \ldots, \tilde{v}_{p^2}^*)$$

$$= (\tilde{v}_{11}, \ldots, \tilde{v}_{pp}, v_{12}, \ldots, v_{1p}, \ldots, v_{p1}, \ldots, v_{p\,p-1}), \qquad (5.12)$$

and A_0 is a function of λ, and p. Further note that,

$$E(\tilde{\underset{\sim}{v}}^*) = \underset{\sim}{\mu}_0 \qquad (5.13)$$

$$\operatorname{Var}(\tilde{\underset{\sim}{v}}^*) = \Delta, \qquad (5.14)$$

where

$$\underset{1\times p^2}{\underset{\sim}{\mu}_\theta'} = (0_1,\ldots,0_p;\ 0,\ldots,0), \tag{5.15}$$

and Δ is given by (4.14) and (5.1).

Since $\lambda_i = \lambda + O(\frac{1}{\sqrt{n_0}})$,

$$\Delta \xrightarrow[n_0\to\infty]{} D_0 = \mathrm{Diag}(2\lambda^2,\ldots,2\lambda^2;\lambda^2,\ldots,\lambda^2). \tag{5.16}$$

The limiting distribution of $\tilde{V}$ is that of $Z = (z_{ij})$ where $Z_{ii} \sim N(\theta_i, 2\lambda^2)$, $Z_{ij} \sim N(0,\lambda^2)$, $i \neq j$ and $Z_{ij}(i \leq j)$'s are all independently distributed. Hence, for large n

$$\underset{\sim}{\tilde{v}}^* \sim N_{p^2}(\mu_\theta,\ D_0), \tag{5.17}$$

where μ_θ and D_0 are given by (5.15) and (5.16) respectively.

Hence

$$\phi_2(t) = \frac{1}{(\sqrt{2\pi})^{p^2}|D_0|^{1/2}} \frac{(r_2-r_1)}{p\lambda} \frac{it}{\sqrt{n_0}} \int e^{it/2\lambda^2 \tilde{v}^{*\prime} A_0 \tilde{v}^* - \frac{1}{2}(\tilde{v}^*-\mu_\theta)' D_0^{-1}(\tilde{v}^*-\mu_0)} \underset{\sim}{c}'\underset{\sim}{\tilde{v}}^*\ d\underset{\sim}{\tilde{v}}^*$$

where

$$\underset{1\times p^2}{\underset{\sim}{c}'} = (p_2\underset{\sim}{\varepsilon}'_{p_1}, -p_1\underset{\sim}{\varepsilon}'_{p_2}, 0\,\underset{\sim}{\varepsilon}'_{p^2-p}) \tag{5.18}$$

$$= \frac{(r_2-r_1)}{p\lambda} \frac{it}{\sqrt{n_0}} \left|I - \frac{it}{\lambda^2} D_0A_0\right|^{-1/2} \exp[-\tfrac{1}{2}(\underset{\sim}{\mu}_\theta' D_0^{-1} \underset{\sim}{\mu}_\theta - \underset{\sim}{v}_\theta'\Omega^{-1}\underset{\sim}{v}_\theta)]\underset{\sim}{c}'\underset{\sim}{v}_\theta \tag{5.19}$$

where

$$\Omega = (D_0^{-1} - \frac{it}{\lambda^2} A_0)^{-1} \tag{5.20}$$

$$\underset{\sim}{v}_\theta = \Omega D_0^{-1} \underset{\sim}{\mu}_\theta$$

$$= \sum_{r=0}^{\infty} \frac{(it)^r}{\lambda^{2r}} (D_0A_0)^r\ \underset{\sim}{\mu}_\theta \tag{5.21}$$

and $\underset{\sim}{c}$ is given by (5.18). Note that

$$(\underset{\sim}{\mu}_0' D_0^{-1} \underset{\sim}{\mu}_0 - \underset{\sim}{v}_0' \Omega^{-1} \underset{\sim}{v}_0)$$

$$= \underset{\sim}{\mu}_0' \left[\sum_{r=1}^{\infty} \frac{(it)^r}{\lambda^{2r}} (\Lambda_0 D_0)^r \right] D_0^{-1} \underset{\sim}{\mu}_0. \tag{5.22}$$

Using (5.21) and (5.22) in (5.19) we have

$$\phi_2(t) = \frac{(r_2 - r_1)}{p\lambda \sqrt{n_0}} it \left| I - \frac{it}{\lambda^2} D_0 \Lambda_0 \right|^{-1/2} \exp\{- \frac{1}{2} \mu_0' (\sum_{r=1}^{\infty} \frac{(it)^r}{\lambda^{2r}} (A_0 D_0)^r) D_0^{-1} \mu_0 \}$$

$$\underset{\sim}{c}' (\sum_{r=0}^{\infty} \frac{(it)^r}{\lambda^{2r}} (D_0 A_0)^r \underset{\sim}{\mu}_0).$$

6. THE DISTRIBUTION OF THE LRT-LIKE TEST STATISTIC WHEN THE JOINT DISTRIBUTION OF OBSERVATIONS IS ELLIPTICALLY CONTOURED

We will first discuss briefly elliptically contoured distributions.

If the random vector $\underset{\sim}{x}$: $p \times 1$ has the characteristic function of the form $\exp(i\, \underset{\sim}{t}' \underset{\sim}{\mu})\ \psi\ (\underset{\sim}{t}' \Sigma\, \underset{\sim}{t})$ where $\underset{\sim}{\mu}$ and $\underset{\sim}{t}$ are of order $p \times 1$, then $\underset{\sim}{x}$ is said to be distributed as elliptically contoured distribution and is denoted by $EC(\underset{\sim}{\mu}, \Sigma; \phi)$. Various properties of elliptically contoured distributions are discussed in Anderson and Fang (1982), Cambanis, Huang and Simons (1981) and Kelker (1970). Now, let

$$E = (\underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_n) = \begin{pmatrix} \underset{\sim}{e}'_{(1)} \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{e}'_{(n)} \end{pmatrix}$$

where $\underset{\sim}{e}^{*\prime} = (\text{vec} E')' = (\underset{\sim}{e}'_{(1)}, \ldots, \underset{\sim}{e}'_{(n)})$. We assume that $\underset{\sim}{e}^* \sim E_{np}(\underset{\sim}{0}, I_n \otimes \Sigma^*; \phi)$ and Σ^* is proportional to Σ given by (4.1). Then, it is known (see Cambanis, Huang and Simons(1981) and

Anderson and Fang (1982)) that

$$E \overset{d}{=} R\,U\,A,$$

where $A'A = \Sigma^*$, $A : p \times p$, $U : n \times p$, Vec. $U = \underset{\sim}{u}^{(np)}$, distribution function of R is related to ϕ and R is independent of U; here $\underset{\sim}{u}^{(np)}$ is np-dimensional column vector which has uniform distribution on the unit sphere. In addition, "$X \overset{d}{=} Y$" denotes that the distribution of X is the same as that of Y. Let us

write $A = (\underset{n\times p_1}{A_1} \quad \underset{n\times p_2}{A_2})$, $p = p_1 + p_2$. Then we have $E_1 \overset{d}{=} R\,U\,A_1$, $E_2 \overset{d}{=} R\,U\,A_2$.

Since

$$S = \begin{pmatrix} E_1'Q_1E_1 & E_1'Q_1Q_2E_2 \\ E_2'Q_2Q_1E_1 & E_2'Q_2E_2 \end{pmatrix} \tag{6.1}$$

we get

$$\begin{aligned} \mathrm{tr}S &= \mathrm{tr}(E_1'Q_1E_1 + E_2'Q_2E_2) \\ &\overset{d}{=} R^2\mathrm{tr}(A_1'U'Q_1UA_1 + A_2'U'Q_2UA_2) \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} |S| &= |E_1'Q_1E_1|\,|E_2'Q_2E_2 - E_2'Q_2Q_1E_1(E_1'Q_1E_1)^{-1}E_1'Q_1Q_2E_2| \\ &\overset{d}{=} R^{2p}|A_1'U'Q_1UA_1|\,|A_2'U'Q_2UA_2 - A_2'U'Q_2Q_1UA_1(A_1'U'Q_1UA_1)^{-1}\,A_1'U'Q_1Q_2UA_2| \end{aligned} \tag{6.3}$$

Hence

$$\frac{|S|}{(\mathrm{tr}S)^p} \overset{d}{=} \frac{|A_1'U'Q_1UA_1|\,|A_2'U'Q_2UA_2 - A_2'U'Q_2Q_1UA_1(A_1'U'Q_1UA_1)^{-1}A_1'U'Q_1Q_2UA_2|}{[\mathrm{tr}(A_1'U'Q_1UA_1 + A_2'U'Q_2UA_2)]^p} \tag{6.4}$$

Substituting this we see that Λ is independent of R^2. Hence the distribution of Λ will be the same as in the normal case. Thus the asymptotic null and nonnull distribution of Λ under assumption (6.1) is the same as in earlier sections.

7. MOMENTS OF THE ESTIMATE OF THE COVARIANCE MATRIX WHEN OBSERVATIONS ON EACH VARIABLE ARE ELLIPTICALLY CONTOURED

In this section, we assume, instead of (6.1), that the covariance matrix of $\underset{\sim}{e}'$ is $\Sigma \otimes I$ where Σ is given by (4.1), $\underset{\sim}{e}' = (\underset{\sim}{e}_1', \ldots, \underset{\sim}{e}_p')$ and

$$\underset{\sim}{e}_j \sim EC_n(0,\ \lambda_j^* I_n;\ \phi) \tag{7.1}$$

where $\lambda_j^* \propto \lambda_j$, $j = 1, \ldots p$.

It can be verified easily that

$$E(\underset{\sim}{e}_i' A \underset{\sim}{e}_i) = -2\phi'(0)\ \lambda_i^* \mathrm{tr} A \tag{7.2}$$

$$E(\underset{\sim}{e}_i' A \underset{\sim}{e}_j) = 0,\ i \neq j \tag{7.3}$$

$$\mathrm{Var}(\underset{\sim}{e}_i' A \underset{\sim}{e}_i) = 8\ \phi'(0)^2 \lambda_i^{*2} \mathrm{tr}\ A^2 + 12(\phi''(0) - \phi'(0)^2)\lambda_i^{*2} \sum_{i=1}^{n} a_{ii}^2 \tag{7.4}$$

$$\text{for } A = A' = (a_{ij})$$

$$\mathrm{Var}(\underset{\sim}{e}_i' A \underset{\sim}{e}_j) = 4\ \lambda_i^* \lambda_j^* [\phi''(0) \sum_{1}^{n} a_{ii}^2 + \phi'(0)^2 \sum_{i \neq j}\sum a_{ij}^2],\ i \neq j. \tag{7.5}$$

Note that $S = \hat{E}'\hat{E}$

$$= \left(\begin{array}{ccc|ccc}
\underset{\sim}{e}_1' Q_1 \underset{\sim}{e}_1 & \cdots & \underset{\sim}{e}_1' Q_1 \underset{\sim}{e}_{p_1} & \underset{\sim}{e}_1' Q_1 Q_2 \underset{\sim}{e}_{p_1+1} & \cdots & \underset{\sim}{e}_1' Q_1 Q_2 \underset{\sim}{e}_p \\
\vdots & & & \vdots & & \\
\underset{\sim}{e}_{p_1}' Q_1 \underset{\sim}{e}_1 & \cdots & \underset{\sim}{e}_{p_1}' Q_1 \underset{\sim}{e}_{p_1} & \underset{\sim}{e}_{p_1}' Q_1 Q_2 \underset{\sim}{e}_{p_1+1} & \cdots & \underset{\sim}{e}_{p_1}' Q_1 Q_2 \underset{\sim}{e}_p \\
\hline
\underset{\sim}{e}_{p_1+1}' Q_2 Q_1 \underset{\sim}{e}_1 & \cdot & \underset{\sim}{e}_{p_1+1}' Q_2 Q_1 \underset{\sim}{e}_{p_1} & \underset{\sim}{e}_{p_1+1}' Q_2 \underset{\sim}{e}_{p_1+1} & \cdots & \underset{\sim}{e}_{p_1+1}' Q_2 \underset{\sim}{e}_p \\
\vdots & & & & & \\
\underset{\sim}{e}_p' Q_2 Q_1 \underset{\sim}{e}_1 & \cdots & \underset{\sim}{e}_p' Q_2 Q_1 \underset{\sim}{e}_{p_1} & \underset{\sim}{e}_p' Q_2 \underset{\sim}{e}_{p_1+1} & \cdots & \underset{\sim}{e}_p' Q_2 \underset{\sim}{e}_p
\end{array}\right)$$

Hence we have

$$E(s_{ii}) = \begin{cases} -2\phi'(0)\lambda_i^* n_1, & i = 1, 2, \ldots, p_1 \\ -2\phi'(0)\lambda_i^* n_2, & i = p_1+1, \ldots, p \end{cases} \tag{7.6}$$

where

$$n_1 = n - r_1,\ n_2 = n - r_2$$

$$E(s_{ij}) = 0,\ \ i \neq j. \tag{7.7}$$

$$\mathrm{Var}(s_{ii}) = \begin{cases} 8\phi'(0)^2\lambda_i^{*2}n_1 + 12(\phi''(0)^2)\lambda_i^{*2}\sum_{i=1}^{n} q_{ii}^{(1)2}, \\ \qquad i = 1,2,\ldots,p_1 \\ 8\phi'(0)^2\lambda_i^{*2}n_2 + 12(\phi''(0)-\phi'(0))^2\lambda_i^{*2}\sum_{i=1}^{n} q_{ii}^{(2)2}, \\ \qquad i = p_1+1,\ldots,p \end{cases} \tag{7.8}$$

where $Q_1 = (q_{ij}^{(1)})$, $Q_2 = (q_{ij}^{(2)})$

$$\mathrm{Var}(s_{ij}) = \begin{cases} 4\lambda_i^*\lambda_j^*[\phi''(0)\sum_\alpha q_{\alpha\alpha}^{(1)2} + \phi'(0)^2\sum\sum_{\alpha\neq\beta} q_{\alpha\beta}^{(1)2} \\ \qquad i,j = 1,2,\ldots,p_1;\ i\neq j \\ 4\lambda_i^*\lambda_j^*[\phi''(0)\sum_\alpha q_{\alpha\alpha}^{(2)2} + \phi'(0)^2\sum\sum_{\alpha\neq\beta} q_{\alpha\beta}^{(2)2}] \\ \qquad i,j = p_1+1,\ldots,p;\ i\neq j \\ 4\lambda_i^*\lambda_j^*[\phi''(0)\sum_\alpha q_{\alpha\alpha}^{(3)2} + \phi'(0)\sum\sum_{\alpha\neq\beta} q_{\alpha\beta}^{(3)2}] \\ \qquad i = 1,2,\ldots,p_1;\ j = p_1+1,\ldots,p \\ \qquad \text{or } i = p_1+1,\ldots,p;\ j = 1,2,\ldots,p_1 \end{cases} \tag{7.9}$$

where $Q_1Q_2 = (q_{ij}^{(3)})$

$$\mathrm{Cov}(s_{ii},s_{jj}) = \begin{cases} 4\lambda_i^*\lambda_j^*(\phi''(0) - \phi'(0)^2)\sum_\alpha q_{\alpha\alpha}^{(1)2} \\ \qquad i,j = 1,2,\ldots,p_1;\ i\neq j \\ 4\lambda_i^*\lambda_j^*(\phi''(0)-\phi'(0)^2)\sum_\alpha q_{\alpha\alpha}^{(2)2} \\ \qquad i,j = p_1+1,\ldots,p;\ i\neq j \\ 4\lambda_i^*\lambda_j^*(\phi''(0)-\phi'(0)^2)\sum_\alpha q_{\alpha\alpha}^{(1)}q_{\alpha\alpha}^{(2)} \\ \qquad i = 1,2,\ldots,p_1,\ j = p_1+1,\ldots,p. \end{cases} \tag{7.10}$$

For simplicity of notation, let

$$r_1 = r_2 = r \Rightarrow n_1 = n_2 = n_0, \quad \text{say}$$

where $n_0 = n-r$.

Hence

$$E(\frac{S}{n_0}) = \mathrm{Diag}(\lambda_1,\ldots,\lambda_p) \tag{7.11}$$

where $\lambda_i = -2\phi'(0)\lambda_i^*$, $i = 1,2,\ldots,p$.

Let us define

$$Z = \sqrt{n_0}\,(\frac{S}{n_0} - D_\lambda). \tag{7.12}$$

Then

$$E(Z) = \underset{p\times p}{0} \tag{7.13}$$

$$\mathrm{Var}(z_{ii}) = \begin{cases} 8\phi'(0)^2\lambda_i^{*2} + 12(\phi''(0)-\phi'(0)^2)\lambda_i^{*2}\sum_i^n q_{ii}^{(1)2}/n_0 \\ \qquad i = 1,2,\ldots,p_1 \\ 8\phi'(0)^2\lambda_i^{*2} + 12(\phi''(0)-\phi'(0)^2)\lambda_i^{*2}\sum_1^n q_{ii}^{(2)2}/n_0 \\ \qquad i = p_1+1,\ldots,p \end{cases} \tag{7.14}$$

$$\mathrm{Var}(z_{ij}) = \begin{cases} 4\lambda_i^*\lambda_j^*[\phi''(0)\sum_\alpha q_{\alpha\alpha}^{(1)2}/n_0 + \phi'(0)^2\sum\sum_{\alpha\neq\beta} q_{\alpha\beta}^{(1)2}/n_0] \\ \qquad i,j = 1(1)p_1;\ i\neq j \\ 4\lambda_i^*\lambda_j^*[\phi''(0)\sum_\alpha q_{\alpha\alpha}^{(2)2}/n_0 + \phi'(0)^2\sum\sum_{\alpha\neq\beta} q_{\alpha\beta}^{(1)2}/n_0] \\ \qquad i,j = p_1+1,\ldots,p;\ i\neq j \\ 4\lambda_i^*\lambda_j^*[\phi''(0)\sum_\alpha q_{\alpha\alpha}^{(3)2}/n_0 + \phi'(0)^2\sum\sum_{\alpha\neq\beta} q_{\alpha\beta}^{(3)2}/n_0] \\ \qquad i = 1(1)p_1;\ j = p_1+1,\ldots,p \\ \qquad \text{or } i = p_1+1,\ldots,p;\ j = 1(1)p_1 \end{cases} \tag{7.15}$$

$$\mathrm{Cov}(z_{ii},z_{jj}) = \begin{cases} 4\lambda_i^*\lambda_j^*(\phi''(0)-\phi'(0)^2)\sum_\alpha q_{\alpha\alpha}^{(1)2}/n_0 & \\ \qquad i,j=1,2,\ldots,p_1;\ i\neq j & \\ 4\lambda_i^*\lambda_j^*(\phi''(0)-\phi'(0)^2)\sum_\alpha q_{\alpha\alpha}^{(2)2}/n_0 & \\ \qquad i,j=p_1+1,\ldots,p;\ i\neq j & \\ 4\lambda_i^*\lambda_j^*(\phi''(0)-\phi'(0)^2)\sum_\alpha q_{\alpha\alpha}^{(1)}q_{\alpha\alpha}^{(2)}/n_0 & \\ \qquad i=1(1)p_1;\ j=p_1+1,\ldots,p. & \end{cases} \tag{7.16}$$

All other elements of Z are uncorrelated. Now make the following assumptions on the design matrices Q_1,Q_2.

Each of $\sum_{\alpha=1}^{n} q_{\alpha\alpha}^{(1)}q_{\alpha\alpha}^{(2)}/n_0$, $\sum_{\alpha=1}^{n} q_{\alpha\alpha}^{(j)2}/n_0$, $j=1,2,3$ and $\sum\sum_{\alpha\neq\beta} q_{\alpha\beta}^{(j)2}/n_0$, $j=1,2,3$ are of $O(1)$ and we write for large n

$$\left.\begin{aligned} &\sum_\alpha q_{\alpha\alpha}^{(j)2}/n_0 = K_1^{(j)}, && j=1,2,3 \\ &\sum\sum_{\alpha\neq\beta} q_{\alpha\beta}^{(j)2}/n_0 = K_2^{(j)}, && j=1,2,3 \\ \text{and} \quad &\sum_\alpha q_{\alpha\alpha}^{(1)}q_{\alpha\alpha}^{(2)}/n_0 = K_3 . && \end{aligned}\right\} \tag{7.17}$$

Note that the limiting distribution of Z is the same as that of $\bar{Z} = (\bar{z}_{ij})$, where

$$\bar{z}_{ii} \sim N(0,2\lambda_i^2\phi), \tag{7.18}$$

$$\bar{z}_{ij} \sim N(0,\lambda_i\lambda_j\psi),\ i\neq j \tag{7.19}$$

where

$$\phi = \begin{cases} 1 + \frac{3}{2}(\phi''(0)/\phi'(0)^2 - 1)K_1^{(1)} & i = 1,2,\ldots,p_1 \\ 1 + \frac{3}{2}(\phi''(0)/\phi'(0)^2 - 1)K_1^{(2)} & i = p_1+1,\ldots,p \end{cases} \tag{7.20}$$

and

$$\psi = \begin{cases} [K_2^{(1)} + \frac{\phi''(0)}{\phi'(0)^2} K_1^{(1)}], & i,j = 1(1)p_1 \\ [K_2^{(2)} + \frac{\phi''(0)}{\phi'(0)^2} K_1^{(2)}], & i,j = p_1+1,\ldots,p \\ [K_2^{(3)} + \frac{\phi''(0)}{\phi'(0)^2} K_1^{(3)}], & i = 1(1)p,\ j = p_1+1,\ldots,p \\ & j = 1(1)p_1,\ i = p_1+1,\ldots,p. \end{cases} \tag{7.21}$$

Also,

$$\operatorname{Cov}(\bar{z}_{ii}, \bar{z}_{jj}) = \lambda_i \lambda_j (\frac{\phi''(0)}{\phi'(0)^2} - 1)C, \quad i \neq j \tag{7.22}$$

where

$$C = \begin{cases} K_1^{(1)}, & i,j = 1(1)p_1 \\ K_1^{(2)}, & i,j = p_1+1,\ldots,p \\ K_3, & i = 1(1)p_1;\ j = p_1+1,\ldots,p. \end{cases} \tag{7.23}$$

All other elements of $\bar{Z}$ are uncorrelated.

It is known that, all fourth order cumulants, if they exist, can be expressed as a function of a single parameter, k, which characterizes the kurtosis of the distribution.

For Σ as in (4.1), the only nonzero fourth order cumulants of the elements of E are

$$K_4^i = E(e_{\ell i}^4) - 3\,\lambda_i^2 \qquad \ell = 1(1)n$$

$$= 3\lambda_i^2\left(\frac{\phi''(0)}{\phi'(0)^2} - 1\right), \qquad i = 1(1)p$$

$$k_{22}^{ij} = E(e_{\ell i}^2 e_{\ell j}^2) - E(e_{\ell i}^2)E(e_{\ell j}^2)$$

$$= \lambda_i\lambda_j\left(\frac{\phi''(0)}{\phi'(0)} - 1\right), \qquad \ell = 1(1)n,$$

$$i,j = 1(1)p,\ i \neq j.$$

All other fourth order cumulants of e_{ij}'s vanish. Note that K_4^i/λ_i^2, is the kurtosis of the marginal distribution of i^{th} component, and for convenience define $K_4^i/\lambda_i^2 = 3k$, $i = 1(1)p$; where k characterizes the kurtosis of the distribution. It is clear from above that $k = \left(\frac{\phi''(0)}{\phi'(0)^2} - 1\right)$, so that

$$K_{22}^{i,j} = \lambda_i\lambda_j k.$$

The parameters in the asymptotic distribution of Z can be expressed as functions of k and λ_i's only.

8. ASYMPTOTIC NULL DISTRIBUTION OF THE LRT-LIKE TEST STATISTIC WHEN THE OBSERVATIONS ON EACH VARIABLE ARE ELLIPTICALLY CONTOURED

Let $S_0 = S/n_0$, then from (7.12), we have

$$S_0 = D_\Lambda + Z/\sqrt{n_0}\ . \tag{8.1}$$

The LRT-like test statistic for sphericity is $\Lambda = \left[\frac{|S_0|}{(\mathrm{tr}\ S_0/p)^p}\right]^{n/2}$.

We are interested in the asymptotic distribution of $T = -2\log\Lambda$. Suppose, under H_0, $\lambda_1 = \ldots = \lambda_p = \sigma^2$. Then

$$S_0 = \sigma^2(I+A), \; A = Z/\sigma_2\sqrt{n_0}$$

$$\log|S_0| = p \log \sigma^2 + \log|I+A|$$

$$\log \operatorname{tr} S_0 = \log \sigma^2 + \log p + \log\left(1 + \frac{\operatorname{tr} A}{p}\right).$$

Hence,

$$T = n\left[p \log\left(1 + \frac{\operatorname{tr} A}{p}\right) - \log|I+A|\right],$$

let $\tilde{T} = \frac{n_0}{n} T$, $\frac{n_0}{n} \le 1$, being the correction factor. Then we have

$$\tilde{T} = \frac{1}{2\sigma^4}\left[\operatorname{tr} Z^2 - \frac{(\operatorname{tr} Z)^2}{p}\right] + 0(n_0^{-1/2}). \tag{8.2}$$

So the characteristic function of $\tilde{T}$ is

$$\begin{aligned}\phi(t) &= E[e^{it\tilde{T}}] \\ &= E[e^{it T_0}] + 0(n_0^{-1/2}),\end{aligned} \tag{8.3}$$

where

$$T_0 = \frac{1}{2\sigma^4}\left[\operatorname{tr} Z^2 - \frac{(\operatorname{tr} Z)^2}{p}\right]. \tag{8.4}$$

Let us write this as $\underset{\sim}{z}'A_0\underset{\sim}{z}$, where

$$\underset{1\times p^2}{\underset{\sim}{z}'} = (z_{11},\ldots,z_{pp},z_{12},\ldots,z_{1p};z_{21},\ldots,z_{2p},\ldots,z_{p1},\ldots,z_{p(p-1)}) \tag{8.5}$$

and A_0 is a matrix whose elements depend on p and σ^2. We assume that the errors are distributed as in Section 7. Then, asymptotically,

$$\underset{\sim}{z} \sim N_{p^2}(0, \Omega), \tag{8.6}$$

where $\Omega = (w_{ij})$

w_{ij}'s are given in Section 7, with the restriction that $\lambda_i = \sigma^2$ for $i = 1(1)p$. Hence,

$$\phi(t) = E[e^{it\,\underset{\sim}{z}'A_0\underset{\sim}{z}}] + O(n_0^{-1/2})$$

$$= |I - 2it A_0\Omega|^{-1/2}. \tag{8.7}$$

Inverting the rightside of (8.7), we get

<u>Theorem 8.1</u> The limiting null distribution as $n_0 \to \infty$ of $\tilde{T}$ is that of a linear combination of chi-squares with one degree of freedom and the coefficients depending on the fourth order moments of the observations of the parent population, which are functions of σ^2 and k.

BIBLIOGRAPHY

[1] Anderson, T. W. and Fang, Kai-tai (1982). "On the theory of multivariate elliptically contoured distributions and their applications", Tech. Report No. 54, Department of Statistics, Stanford University, Stanford, CA.

[2] Cambanis, S. Huang, S. and Simons, G. (1981). "On the theory of elliptically contoured distributions", J. Multivariate Anal. 11, 368-385.

[3] Fujikoshi, Y. (1981). "Asymptotic expansions for the distributions of some multivariate tests under local alternatives", Tamkang Journal of Math., 12, 117-136.

[4] Kariya, T., Fujikoshi, Y. and Krishnaiah, P. R. (1983). "Tests for independence of two multivariate regression models with different design matrices". To appear in J. Multivariate Anal.

[5] Kariya, T. and Maekawa, K. (1982). "A method for approximations to the pdf's and cdf's of GLSE's and its application to the seemingly unrelated regressions model", Ann. Inst. Statis. Math., 34, 281-297.

[6] Kelker, D. (1970). "Distribution theory of spherical distributions and a location scale parameter generalization", Sankhya, Ser. A, 32, 419-430.

[7] Lee, J. C., Krishnaiah, P. R. and Chang, T. C. (1977). "Approximations to the distributions of the likelihood ratio statistics for testing certain structures on the covariance matrices of real multivariate normal populations", In Mult. Anal. - IV (P. R. Krishnaiah, Ed.), North-Holland Publishing Co.

[8] Maekawa, K. (1982). "Relations among several asymptotic expansions in seemingly unrelated regression model", The Hiroshima Economic Review, 5, 55-66.

[9] Mauchly, J. W. (1940). "Significance test for sphericity of a normal p-variate distribution", Ann. Math. Statist., 11, 204-209.

[10] Revankar, N. S. (1974). "Some finite sample results in the context of two seemingly unrelated regression equations", J. Amer. Statist. Assoc., 69, 187-190.

[11] Revankar, N. S. (1976). "Use of restriced residuals in SUR systems: some finite sample results", J. Amer Statist. Assoc., 71, 183-187.

[12] Sarkar, Shakuntala, and Krishnaiah, P. R. (1984). "Estimation of parameters under correlated regression equations model", Tech. Report No. 84-21, Center for Multivariate Analysis, University of Pittsburgh.

[13] Srivastava, J. N. (1966). "Some generalizations of multivariate analysis of variance". In Multivariate Analysis (P.R. Krishnaiah, editor). Academic Press, New York.

[14] Srivastata, V. K. (1970). "The efficiency of estimating seemingly unrelated regression equations", Ann. Inst. Statist. Math., 22, 483-493.

[15] Srivastava, V. K. (1973). "The efficiency of an improved method of estimating seemingly unrelated regression equations", J. Econometrics, 3, 341-350.

[16] Trawinski, I. M. (1961). Incomplete Variable Designs. Ph.D. Thesis. Virginia Polytechnic Institute, Blacksburg, Virginia.

[17] Zellner, Arnold (1962). "An efficient method of estimating seemingly unrelated regressions and tests of aggregation bias", J. Amer. Statist. Assoc., 57, 348-368.

[18] Zellner, Arnold (1963). "Estimators of seemingly unrelated regression equations: some exact finite sample results", J. Amer. Statist. Assoc., 58, 977-992.

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